

Higher \hat{A} -genera on certain non-spin S^1 -manifolds

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Abstract

We prove the vanishing of higher \hat{A} -genera, in the sense of Browder and Hsiang [4], on smooth S^1 -manifolds with finite π_2 and π_4 .

Keywords: \hat{A} -genus, circle actions, elliptic genus.

1 Introduction

The classical result of Atiyah and Hirzebruch AH about the vanishing of the \hat{A} -genus on Spin manifolds with S^1 actions was generalized by Browder and Hsiang Browder to higher \hat{A} -genera in the following form.

Theorem 1.1 [4, Theorem 1.8] *Let M be a closed Spin manifold with a smooth effective action of a compact, connected, positive-dimensional Lie group G . Then*

$$p_*([M] \cap \hat{A}) = 0,$$

where $p: M \rightarrow M/G$, and $\hat{A} \in H^{4*}(M; \mathbb{Q})$ is the \hat{A} polynomial in the Pontrjagin classes.

Furthermore, from this theorem they also deduced a higher \hat{A} -genus theorem analogous to Novikov's "higher signature".

By a closed manifold M , we mean a compact manifold without boundary. Notice that if G is a compact Lie group not necessarily connected then we restrict our attention to the connected component of the identity element.

In this paper, we prove two theorems (Theorems 1.2 and 4.1) for non-Spin G -manifolds with finite π_2 and π_4 . They are analogous to those of Browder and Hsiang [4] for Spin manifolds.

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Theorem 1.2 *Let M be a smooth, closed, connected, oriented (even-dimensional) G -manifold with finite $\pi_2(M)$ and $\pi_4(M)$, where G is a compact, connected, positive-dimensional Lie group. Then for any $y \in H^*(M/G, \mathbb{Q})$*

$$(\hat{A} \cup p^*(y))[M] = 0,$$

which implies

$$p_*([M] \cap \hat{A}) = 0,$$

where $p: M \rightarrow M/G$ is the projection map, $\hat{A} \in H^{4}(M; \mathbb{Q})$ is the \hat{A} polynomial.*

The proof will make use of the G -transversality approach of Browder and Quinn [5], properties of G -transverse submanifolds, and the rigidity of the elliptic genus on manifolds admitting 2-balanced S^1 actions (see below).

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2 S^1 -transverse submanifolds of manifolds with finite π_2 and π_4

Definition 2.1 *Let G be a connected Lie group acting smoothly on a manifold M . Let H be a subgroup of G . We denote by M^H the fixed point set of H on M . A G -invariant submanifold N of M is called *transverse* if N intersects M^H transversely for every subgroup H of G .*

In order to prove Theorem 1.2, we only need to consider a circle action. Thus, we can choose any circle subgroup $S^1 \subseteq G$. We denote by M^{S^1} the fixed point set of the circle action. At a fixed point $p \in M^{S^1}$, the tangent space of M becomes a real representation of S^1 , whose complexification can be written as

$$T_p M \otimes \mathbb{C} = (t^{m_1} + t^{-m_1}) + \cdots + (t^{m_d} + t^{-m_d})$$

where t^a denotes the representation on which $\lambda \in S^1$ acts by multiplication by λ^a , and d is half the dimension of M . The term $(t^n + t^{-n})$ corresponds to the representation

$$\lambda = e^{i\theta} \in S^1 \mapsto \begin{pmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{pmatrix}.$$

The numbers $\pm m_1, \dots, \pm m_d$ are called the exponents (or weights) of the S^1 -action at the point p . A circle action is called *2-balanced* if the parity of $\sum_{i=1}^d m_i$ does not depend on the connected component of M^{S^1} (cf. HiBJ). Since we are only interested in the parity of $\sum_{i=1}^d m_i$, we do not worry about the choice of signs.

Lemma 2.2 *Let M be a S^1 -manifold with finite $\pi_2(M)$ and $\pi_4(M)$. Let N be an S^1 -transverse submanifold of M . Then the S^1 action on N is 2-balanced.*

Proof. Since N meets M^{S^1} transversely, for $p \in N \cap M^{S^1}$

$$T_p M = T_p N + T_p M^{S^1}.$$

Notice that $N^{S^1} = N \cap M^{S^1}$. Let $p, p' \in N^{S^1}$ lie in two different components of N^{S^1} . The tangent spaces to N at p and p' become S^1 representations so that

$$\begin{aligned} T_p N \otimes \mathbb{C} &= (t^{n_1(p)} + t^{-n_1(p)}) + \dots + (t^{n_k(p)} + t^{-n_k(p)}), \\ T_{p'} N \otimes \mathbb{C} &= (t^{n_1(p')} + t^{-n_1(p')}) + \dots + (t^{n_k(p')} + t^{-n_k(p')}), \end{aligned}$$

and we have to verify that

$$(n_1(p) + \dots + n_k(p)) - (n_1(p') + \dots + n_k(p')) \equiv 0 \pmod{2}. \quad (1)$$

Observe that the numbers $n_i(p)$ and $n_i(p')$ are, in fact, exponents of the action of S^1 on the manifold M , and that (1) is the difference of exponents of the manifold M , since the only missing directions of the tangent space of M are trivial representations (as N and M^{S^1} meet transversely). Since M has finite $\pi_2(M)$ and $\pi_4(M)$, by [3, Theorem V]

$$f(t) = T_p M_c - T_{p'} M_c = (1 - t)^3 P(t),$$

where $P(t) = \sum b_i t^i$ with only finitely many b_i 's different from zero. Since real representations are invariant under the automorphism $t \mapsto t^{-1}$

$$f(t) = f(t^{-1}),$$

i.e.

$$(1 - t)^3 P(t) = \left(1 - \frac{1}{t}\right)^3 P(t^{-1}).$$

Thus,

$$\begin{aligned} t^3 P(t) &= -P(t^{-1}), \\ t^3 P(t) + P(t^{-1}) &= 0, \\ t^3 \sum b_i t^i + \sum b_i t^{-i} &= 0, \\ t^{3/2} \sum b_i (t^{i+3/2} + t^{-i-3/2}) &= 0. \end{aligned}$$

Since $b_i \neq 0$, for every term of the form $b_i(t^{i+3/2} + t^{-i-3/2})$ there must be another one that cancels it out, i.e. there must be a $j \neq i$ such that $b_i = -b_j$ so that either $j + 3/2 = i + 3/2$ which cannot happen because it contradicts $i \neq j$, or $-j - 3/2 = i + 3/2$, and $i = -3 - j$. Then, all the terms of $P(t)$ can be grouped according to the corresponding pairs

$$b_i t^i + b_j t^j = b_i t^i - b_i t^{-3-i},$$

which multiplied by $(1 - t)^3$ give

$$\begin{aligned} (b_i t^i - b_i t^{-3-i})(1 - t)^3 &= b_i(t^i + t^{-i}) - 3b_i(t^{i+1} + t^{-(i+1)}) \\ &+ 3b_i(t^{i+2} + t^{-(i+2)}) - b_i(t^{i+3} + t^{-(i+3)}). \end{aligned}$$

Taking the sum of the exponents (with any choice of signs) with multiplicity gives zero (mod 2),

$$b_i(i) - 3b_i(i + 1) + 3b_i(i + 2) - b_i(i + 3) \equiv 0 \pmod{2}.$$

Remark 2.3 Note that the lemma is still valid if we only require the S^1 action on M to be 2-balanced instead of M having finite $\pi_2(M)$ and $\pi_4(M)$.

3 Elliptic genus on manifolds with 2-balanced S^1 -actions

Let \bigwedge_c^\pm be the even and odd complex differential forms on the oriented, closed, smooth manifold X under the Hodge $*$ -operator, respectively. The signature operator

$$d_s^X = d - *d* : \bigwedge_c^+ \longrightarrow \bigwedge_c^-$$

is elliptic and the virtual dimension of its index equals the signature of X , $\text{sign}(X)$. If W is a complex vector bundle on X endowed with a connection, we can *twist* the signature operator to forms with values in W

$$d_s^X \otimes W : \bigwedge_c^+(W) \longrightarrow \bigwedge_c^-(W).$$

This operator is also elliptic and the virtual dimension of its index is denoted by $\text{sign}(X, W)$.

Definition 3.1 Let $T = TX \otimes \mathbb{C}$ denote the complexified tangent bundle of X and let R_i be the sequence of bundles defined by the formal series

$$R(q, T) = \sum_{i=0}^{\infty} R_i q^i = \bigotimes_{i=1}^{\infty} \bigwedge_{q^i} T \otimes \bigotimes_{j=1}^{\infty} S_{q^j} T,$$

where $S_t T = \sum_{k=0}^{\infty} S^k T t^k$, $\bigwedge_t T = \sum_{k=0}^{\infty} \bigwedge^k T t^k$, and $S^k T$, $\bigwedge^k T$ denote the k -th symmetric and exterior tensor powers of T , respectively. The elliptic genus of X is defined as

$$\Phi(X) = \text{ind}(d_s^X \otimes R(q, T)) = \sum_{i=0}^{\infty} \text{sign}(X, R_i) \cdot q^i. \quad (2)$$

Note that the first few terms of the sequence $R(q, T)$ are $R_0 = 1$, $R_1 = 2T$, $R_2 = 2(T^{\otimes 2} + T)$. In particular, the constant term of $\Phi(X)$ is $\text{sign}(X)$.

If we assume that G is a group acting on M and commuting with the elliptic operator, then for $g \in G$ the equivariant index of D can be defined as

$$\text{index}(D)_G(g) = \text{trace}(g, \text{Ker} D) - \text{trace}(g, \text{Coker} D).$$

In an analogous way to the definition of the elliptic genus, now we define the equivariant elliptic genus with respect to the S^1 action by

$$\Phi(X)_{S^1}(\lambda) = \sum_{i=0}^{\infty} \text{sign}(X, R_i)_{S^1}(\lambda) \cdot q^i, \quad (3)$$

where $\lambda \in S^1$.

Theorem 3.2 *Let X be an $2n$ -dimensional, oriented, closed, smooth manifold admitting a smooth 2-balanced S^1 -action. Then*

$$\Phi(X) = \Phi(X)_{S^1}(\lambda) \quad (4)$$

for every $\lambda \in S^1$.

Sketch of proof. The proof of Theorem 3.2 is along the lines of BT. The equivariant elliptic genus $\Phi(X)_{S^1}(\lambda)$ turns out to be a meromorphic function on $T_{q^2} = \mathbb{C}^*/q^2$ (the non-zero complex numbers modulo the multiplicative group generated by $q^2 \neq 0$). Thus, the proof of the theorem reduces to proving that $\Phi(X)_{S^1}(\lambda)$ has no poles at all on T_{q^2} , thus implying that $\Phi(X)_{S^1}(\lambda)$ is constant in λ . This follows from applying the Atiyah-Segal equivariant index theorem and localizing to the S^1 -fixed point set and other auxiliary submanifolds. More precisely, one can define the translate $t_a \Phi(M)_{S^1}(\lambda)$ of $\Phi(M)_{S^1}(\lambda)$ by $a \in \mathbb{C}^*$, to be given by the map at the character level $\lambda \mapsto a\lambda$. In order to prove the rigidity theorem of $\Phi(M)$, we shall show that none of the translates $t_a \Phi(M)$, $a \in T_{q^2}$, by points of finite order on T_{q^2} , has a pole on the circle $|\lambda| = 1$. The translates $t_a \Phi(M)$ can be expressed as *twists* of the elliptic genus on some auxiliary manifolds. The auxiliary submanifolds are the fixed point sets X_k of the subgroups $\mathbb{Z}_k \subset S^1$, $k \in \mathbb{Z}$. In doing so, the corresponding expressions have no poles at 1, and thus $\Phi(X)_{S^1}(\lambda)$ has no poles at points of finite order in T_{q^2} . This argument is valid as long as:

- (i) the submanifolds X_k containing S^1 -fixed points are orientable;
- (ii) it is possible to choose an orientation of X_k compatible with X and all the components P contained in X_k .

(i) is proved in [6, Lemma 1]. (ii) follows as in [2, Lemma 9.3] but using the fact that the action is 2-balanced.

Corollary 3.3 *Let X be a even-dimensional, oriented, closed, connected, smooth manifold admitting a 2-balanced S^1 action. If the S^1 action is non-trivial then*

$$\widehat{A}(X) = 0.$$

The proof follows in the same way as in [8, Theorem, Section 1.5]. \square

Corollary 3.4 *Let G be a compact positive-dimensional Lie group. Let N be a compact G -transverse submanifold of a connected, oriented, smooth G -manifold with finite $\pi_2(M)$ and $\pi_4(M)$. Then the \widehat{A} -genus of N vanishes*

$$\widehat{A}(N) = 0.$$

This follows from Lemma 2.2 and Corollary 3.3. \square

Remark 3.5 Note that N is not necessarily a Spin manifold since M is not required to be so. Thus, the vanishing of $\widehat{A}(N)$ is not a consequence of the Atiyah-Hirzebruch vanishing theorem.

Remark 3.6 Note that M does not need to be compact for Corollary 3.4 to hold.

4 Vanishing of higher \widehat{A} -genera

In this section we provide the proof of Theorem 1.2. The proof follows that of Theorem 1.8 in [4], and uses the rigidity of the elliptic genus on manifolds admitting 2-balanced S^1 actions. We also prove Theorem 4.1, which can be thought of as a higher \widehat{A} -genus theorem for G -manifolds with finite $\pi_2(M)$ and $\pi_4(M)$.

Proof of Theorem 1.2. We use the G -transversality approach of Browder and Quinn [5]. In it, given a manifold X (not necessarily compact) endowed with an action of G , they establish a 1-1 correspondence between transverse bordism classes of compact framed G -submanifolds of X of codimension k with homotopy classes of maps from X/G^* to the sphere S^k , $[X/G^*, S^k]$. Here X/G^* is the 1-point compactification of X/G .

Given a space Y , we denote by $\Sigma^t Y$ the t -fold reduced suspension of Y , which is homeomorphic to the smash product of Y and S^t , $\Sigma^t Y = Y \wedge S^t$.

To apply [4, Theorem 4.2], let $y \in H^l(M/G)$. Since rational stable cohomology and rational stable cohomotopy are isomorphic, we can find $t \in \mathbb{N}$ and a map

$$\rho : \Sigma^t(M/G_+) \longrightarrow S^{l+t}$$

such that

$$\rho^*(g) = \Sigma^t y,$$

where g generates $H^{l+t}(S^{l+t})$, and M/G_+ is the disjoint union of M/G with a base point. Notice that $\Sigma^t(M/G_+) = (M/G \times \mathbb{R}^t)^*$, the one point compactification of $M/G \times \mathbb{R}^t$. One can consider $M \times \mathbb{R}^t$ as a G -manifold, by extending the action of G to the \mathbb{R}^t factor by a trivial action. By [4, Theorem 4.2] there is a compact transverse framed G -submanifold $i : N \hookrightarrow M \times \mathbb{R}^t$, such that

$$p^* \rho^*(g) \cap [(M \times \mathbb{R}^t)^*] = i_*[N],$$

i.e. the Poincaré dual of $i_*[N]$ is $p^* \rho^*(g)$, which follows from the construction of the submanifold N in the proof of Lemma 4.4 in [4].

Note that

$$\widehat{A}(N) = i^* \widehat{A}(M),$$

where $\widehat{A}(M) \in H^{4*}(M; \mathbb{Q})$. Since $M \times \mathbb{R}^t$ has finite $\pi_2(M \times \mathbb{R}^t)$ and $\pi_4(M \times \mathbb{R}^t)$, and N is a G -transverse submanifold of $M \times \mathbb{R}^t$, the G action on N is non-trivial and $\widehat{A}(N)[N] = 0$. Hence, by Corollary 3.4

$$\begin{aligned} 0 &= \widehat{A}(N)[N] \\ &= (i^* \widehat{A}(M))[M] \\ &= \widehat{A}(M)(i_*[N]) \\ &= \widehat{A}(M)(p^* \rho^* \cap [M \times \mathbb{R}^t]) \\ &= \widehat{A}(M)(p^* \Sigma^t y \cap [M \times \mathbb{R}^t]) \\ &= \widehat{A}(M)(p^* y \cap [M]) \\ &= (\widehat{A}(M) \cup p^* y)[M]. \end{aligned}$$

□

Let $f : M \longrightarrow K(\pi_1(M), 1)$ be a map, assume that $f_* : \pi_1(M) \rightarrow \pi_1(M)$ is onto, one can define π' to be $\pi_1(M)/f_* i_*(\pi_1(G))$, where $i : G \rightarrow M$ is induced by the action of G on the base point of M . Notice that $i_*(\pi_1(G))$ is contained in the center of $\pi_1(M)$ [9, page 40]. Let $\alpha : \pi_1(M) \rightarrow \pi'$ be the projection.

Theorem 4.1 *Let M be a closed, connected, smooth manifold with finite $\pi_2(M)$ and $\pi_4(M)$, and let G be a compact, connected, positive-dimensional Lie group acting smoothly and effectively on M . Let $f : M \longrightarrow K(\pi_1(M), 1)$, and $x \in H^*(K(\pi', 1); \mathbb{Q})$. Then $(f^* \alpha^*(x) \cup \widehat{A})[M] = 0$, where $\widehat{A} \in H^*(M; \mathbb{Q})$ is the \widehat{A} polynomial in the Pontrjagin classes.*

Proof. By Theorem 1.1 in Browder, there is a map $\phi : H_*(M/S^1, \mathbb{Q}) \longrightarrow H_*(K(\pi', 1), \mathbb{Q})$ such that the following diagram commutes,

$$\begin{array}{ccc} H_*(M, \mathbb{Q}) & \xrightarrow{f_*} & H_*(K(\pi_1(M), 1), \mathbb{Q}) \\ \downarrow p_* & & \downarrow \alpha_* \\ H_*(M/S^1, \mathbb{Q}) & \xrightarrow{\phi} & H_*(K(\pi', 1), \mathbb{Q}), \end{array}$$

so that $(\hat{A} \cup f^* \alpha^*(x))[M] = (\hat{A} \cup p^* \phi^*(x))[M] = 0$, for every $x \in H^*(K(\pi', 1), \mathbb{Q})$. □

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